Aeroelastic Optimization of a Panel in High Mach Number Supersonic Flow

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This paper presents a solution for a least-weight skin thickness distribution for a panel with a flutter parameter constraint. This panel weighs less than any similar constant thickness panel, but has the same critical supersonic panel flutter parameter λ_{cr}. The panel rests on simple supports and is of sandwich construction. The span to chord ratio is large enough that the inertial, elastic, and aerodynamic behavior is one-dimensional. The Mach number is great enough that the aerodynamic forces acting on the upper panel surface may be accurately described by quasi-steady, linearized, supersonic aerodynamic theory. The final optimum design is obtained from theoretical and numerical methods adapted from optimal control theory. The results of this investigation show that the optimal panel thickness distribution is symmetric about the panel chord midpoint. Compared to a reference panel with constant thickness, optimum panels are found to be nearly 12% lighter.

Nomenclature

a	= panel chord length
$D(x)$	= panel chordwise bending stiffness
$m(x)$	= panel mass per unit area
M	= Mach number
MR	= mass ratio; total optimum panel weight/total reference panel weight
q_0	= aerodynamic pressure
$t(x)$	= nondimensional thickness parameter; $T(x)/T_0$
$T(x)$	= dimensional panel face-sheet thickness
T_0	= dimensional reference panel face-sheet thickness, a constant
$w^*(x,\tau)$	= nondimensional panel displacement/chord
x	= nondimensional chordwise coordinate
z_0	= nondimensional frequency ratio parameter [Eq. (9a)]
δ_1	= reference panel ratio of face-sheet mass to total mass
ξ	$= 1 - x$
λ_0, λ_{cr}	= aerodynamic parameter, critical parameter for the onset of flutter $\lambda_0 = 2q_0a^3/D_0(M^2-1)^{1/2}$
τ	= time, sec
ω	= vibration frequency; rad/sec
{ }	= column matrix
[ ]	= row matrix
( )′	$= d(\,)/dx$
$(\dot{\,})$	$= d(\,)/d\xi$

Introduction

BECAUSE of the flexibility of aircraft structures, the design of light-weight structural components with specified values of aeroelastic design parameters is of importance. The aeroelastic optimization of a panel whose flutter parameter λ_{cr} is fixed throughout the optimization search has been the subject of several previous investigations. This panel flutter optimization problem was first suggested by Ashley and McIntosh.[1] They suggested a variational calculus approach which leads to a nonlinear differential equation description of the problem. Turner[2] studied the same problem, but used discrete parameter methods to model the panel behavior. He applied his method to a four element panel model and deduced from his equations that the solution for the optimal or least-weight mass distribution should be symmetric about the midchord point. Armand[3] studied the panel problem independently and reached a similar conclusion. No rigorous proof of the uniqueness of the final solution was demonstrated by either Turner or Armand. Thus, there may be a remote possibility that more than one solution to the problem exists. This possibility will be discussed in a later section.

This study uses variational calculus methods to investigate the panel flutter optimization problem. The constraint equations are differential equations which have eigenvalues and specified boundary conditions. The necessary conditions for least weight, subject to these constraints, will be discussed. The existence of a symmetric panel thickness will be used to develop a numerical solution technique. This scheme is adapted from the "transition matrix" algorithm used in optimal control problems.[4] It was chosen because of its accuracy and the simplicity of programing it for the computer.

Constraint Equations for Panel Flutter

As shown in Ref. 5, the partial differential equation for equilibrium of the one-dimensional panel model shown in Fig. 1 may be written, in nondimensional form, as

$$\frac{\partial^2}{\partial x^2}\left[\left(\frac{D(x)}{D_0}\right)\frac{\partial^2 w^*}{\partial x^2}\right] + \lambda_0\left(\frac{\partial w^*}{\partial x}\right) + \lambda_0\left[\left(\frac{M^2-2}{M^2-1}\right)^{1/2}\left(\frac{a}{U}\right)\right]\frac{\partial w^*}{\partial \tau} + R_{xx}\left(\frac{\partial^2 w^*}{\partial x^2}\right) + \left(\frac{m(x)}{m_0}\right)\left(\frac{m_0 a^4}{D_0}\right)\frac{\partial^2 w^*}{\partial \tau^2} = 0 \quad (1)$$

The subscripted quantities $(\,)_0$ refer to reference panel values of elastic, inertial, and aerodynamic parameters. This reference panel has constant core depth and face-sheet thickness. Free vibration oscillations are of the form

$$w^*(x,\tau) = w(x)e^{i\omega\tau} \quad (2)$$

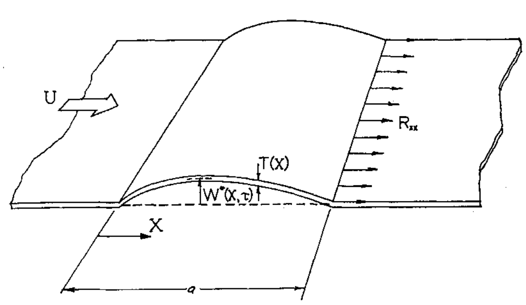

Fig. 1 One-dimensional panel flutter model.

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Equation (1) then may be expressed as a nondimensional ordinary differential equation

$$\frac{d^2}{dx^2}\left[\left(\frac{D(x)}{D_0}\right)\frac{d^2w}{dx^2}\right]+\lambda_0\left(\frac{dw}{dx}\right)+R_{xx}\frac{d^2w}{dx^2}-k\left(\frac{m(x)}{m_0}\right)w(x)=0 \tag{3}$$

the parameter k is, in general, a complex number

$$k=(m_0a^4/D_0)\omega^2-i\lambda_0\omega(M^2-2)/(M^2-1)^{1/2}(a/U) \tag{4}$$

The imaginary term in Eq. (4) is an aerodynamic damping term and is a function of λ_0, ω and the airspeed U. In a great number of cases, this term may be neglected.[5] This analysis will neglect aerodynamic damping because of the simplification which results. The boundary conditions for a panel on simple supports are

$$w(0)=w(1)=D(0)w''(0)=D(1)w''(1)=0 \tag{5}$$

For specific values of $D(x)$ and R_{xx}, a value of λ_0 may be found which will cause some of the frequencies ω to become complex. If the imaginary part of ω is negative, then the motion will be divergent in time. The panel is then unstable. Solutions for panel instability when $D(x)$ equals D_0 are well known.[5] A further simplification can be made if the in-plane stress term R_{xx} is also neglected. For this case, Eq. (3) may be written as

$$\frac{d^2}{dx^2}\left[\left(\frac{D(x)}{D_0}\right)\frac{d^2w}{dx^2}\right]+\lambda_0\left(\frac{dw}{dx}\right)-\pi^4\left[\left(\frac{\omega}{\omega_0}\right)^2\left(\frac{m(x)}{m_0}\right)\right]w(x)=0 \tag{6}$$

where $\omega_0{}^2=D_0\pi^4/m_0a^4$. For a constant-thickness reference panel, the lowest two frequencies "merge" at $\lambda_0=343.2$ and become complex conjugate above $\lambda_0=343.2$.

For a panel of sandwich construction, the bending stiffness $D(x)$ is a linear function of the face-sheet thickness $T(x)$. Then

$$D(x)/D_0=T(x)/T_0=t(x) \tag{7}$$

If a thin, uniform sheet of nonstructural mass is sandwiched between the face-sheets and is not involved in the optimization process, then it can be shown that

$$m(x)/m_0=\delta_1t(x)+\delta_2 \tag{8}$$

The parameters δ_1 and δ_2 are not independent since

$$\delta_2=1-\delta_1$$

The parameter δ_1 is the ratio between the initial or reference face-sheet weight and the total weight of the reference panel. For ease of notation, four new parameters are defined

$$\alpha=\delta_1[\pi^4]\left[\frac{\omega}{\omega_0}\right]^2=(z_0\pi)^4\delta_1 \tag{9a}$$

$$\beta=(1-\delta_1)[\pi^4]\left[\frac{\omega}{\omega_0}\right]^2=(z_0\pi)^4(1-\delta_1) \tag{9b}$$

Equation (6) now becomes

$$(tw'')''+\lambda_0w'-(\alpha t+\beta)w(x)=0 \tag{10}$$

A new set of independent variables can be defined by the following set of first-order differential equations:

$$w'(x)=p(x) \tag{11a}$$
$$p'(x)=q(x)/t(x) \qquad 0\leq x\leq 1 \tag{11b}$$
$$q'(x)=r(x) \tag{11c}$$

Equation (10) can be written as a function of these variables

$$r'(x)=(\alpha t+\beta)w(x)-\lambda_0p(x) \tag{11d}$$

Since the panel free vibration behavior is governed by Eqs. (11a–d), they form a set of constraint equations on the nondimensional thickness parameter $t(x)$. To be considered as a candidate for the least-weight design, a function $t(x)$ must satisfy Eqs. (11a–d) with the simple support boundary conditions. In terms of the new functions

$$w(0)=w(1)=q(0)=q(1)=0 \tag{12}$$

Not only must $t(x)$ satisfy these equations, but the distribution must have a value of λ_{cr} equal to 343.2. To guarantee this last requirement, values of λ_0 and ω (or z_0) must be provided.

Necessary Conditions for Optimum Design

Since the panel has a large spanwise dimension, the terms "minimum mass" or "least weight" may be imprecise. It is better if one speaks of minimizing the total weight of a strip of panel with a spanwise width unity. Then, the quantity to be minimized is

$$J=\int_0^1 t(x)dx \tag{13}$$

J is the "objective function" and represents the ratio of the total weight of the face-sheets of a nonuniform thickness panel to the weight of the face-sheets in a reference panel. The nonstructural mass is kept constant and adds a constant term to the face-sheet weight. For this reason, it is not included in the objective function.

The problem may now be stated. It is to determine the function $t(x)$ which, using the constraint equations, gives $\lambda_{cr}=343.2$ and yields a minimum value of J. An additional constraint which results in a more realistic design is the thickness inequality constraint. This constraint is expressed as

$$t_{min}-t(x)\leq 0 \quad \text{for} \quad 0\leq x\leq 1 \tag{14}$$

The constant t_{min} is the minimum allowable value of $t(x)$. This is equivalent to specifying a minimum gage for the panel. The differential equation constraints (11a–d) for this problem are of the form

$$dg_i/dx=f_i(g_j,x,t) \qquad i=1,2,3,4 \tag{15}$$

In optimal control theory, the variables $g_i(x)$ are known as "state variables" while $t(x)$ is the "control variable." The panel objective function may be written as

$$J=\int_0^1 L(g_i,x,t)\,dx \tag{16}$$

This problem is the Lagrange problem of variational calculus. Optimal control theory solves this problem by adjoining the state variable equations to the objective function J, with Lagrange multipliers $\lambda_i(x)$. This results in a functional, the Hamiltonian, H

$$H=L(g_i,x,t)+[\lambda_i]\{f_i\} \tag{17}$$

For this panel flutter problem, the Hamiltonian is given by

$$H=t(x)+\lambda_wp+\lambda_p(q/t)+\lambda_qr+\lambda_r([\alpha t+\beta]w-\lambda_0p) \tag{18}$$

From control theory, a necessary set of conditions for an extremum of J includes Eqs. (11a–d), the state variable boundary conditions (12) and the following adjoint differential equations[4]:

$$-\partial H/\partial w=\lambda_w'(x)=-\lambda_r(x)[\alpha t(x)+\beta] \tag{19a}$$
$$-\partial H/\partial p=\lambda_p'(x)=-\lambda_w(x)+\lambda_0\lambda_r(x) \tag{19b}$$
$$-\partial H/\partial q=\lambda_q'(x)=-\lambda_p(x)/t(x) \tag{19c}$$
$$-\partial H/\partial r=\lambda_r'(x)=-\lambda_q(x) \tag{19d}$$

with

$$\lambda_p(0)=\lambda_p(1)=\lambda_r(0)=\lambda_r(1)=0 \tag{19e}$$

The control variable equation for $t(x)$, subject to Eq. (14), is

$$t(x) = \begin{cases} t_1(x), \text{ if } t_1^2(x) = \lambda_p(x)q(x)/[1 + \alpha\lambda_r(x)w(x)] > t_{\min}^2 \\ t_{\min}, \text{ if } t_1^2 \leq t_{\min}^2 \end{cases} \quad (19f)$$

Now the panel flutter optimization problem involves the solution of a set of eight nonlinear differential equations with four specified boundary conditions at $x = 0$ and four specified conditions at $x = 1$. In addition, the distribution $t(x)$ must have a critical flutter parameter λ_{cr} equal to 343.2. Several researchers have attempted to find a closed form solution to these equations. As yet, no such solution has been found.

Character of the Optimality Equations

The panel constraint equations (11a–d) are real and depend on two parameters, λ_0 and ω. The actual design constraint, λ_{cr}, does not appear explicitly in these equations. The values of λ_0 and ω must be chosen to ensure that the distribution found from the solution to the optimality equations satisfies the flutter constraint.

Figure 2 shows the behavior of the first two natural frequencies of the reference panel as a function of λ_0. The merging frequency lies between the in vacuo natural frequencies ω_1 and ω_2. An infinite number of panel thickness distributions exist which have the same critical value of λ_0 as the reference panel. Each of these panels has a different value of the objective function and each may have a different merging frequency at λ_{cr}. One of these designs is the desired least-weight panel. However, the optimality criteria ensure only a stationary value of J. Unless uniqueness can be shown, there is a possibility that relative minima or maxima exist.

If one were to determine all designs having the same critical value of λ_0 and differing values of the merging frequency, complex constraint equations and multiplier equations would be necessary. In addition, an additional constraint involving the merging frequency would be necessary. This would double the number of equations and complicate the problem. When instability occurs because of frequency merging, the search process may possibly be simplified if certain assumptions are valid.

Assume that the merging of the first two natural frequencies,

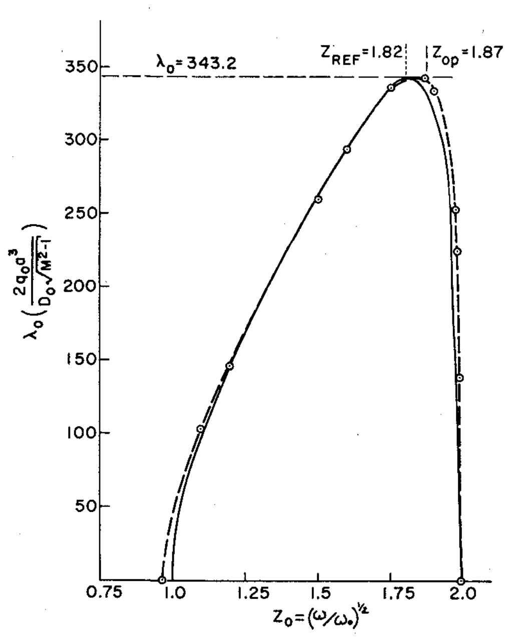

Fig. 2 Typical λ_0 vs z_0 curves. Constant thickness panel shown as solid line. Typical least-weight panel behavior shown as dashed line.

for any panel, has a behavior similar to that of the reference panel. If numerical values of λ_0 and ω, denoted λ_e, and ω_e, are inserted into the optimality equations and the equations are solved, then a value of $t(x)$ may be determined. The panel having this distribution has a λ_0 vs ω behavior associated with it, as well as a merging point and a value of λ_{cr}. Since $t(x)$ satisfies the constraint equations for a given λ_e, ω_e point, this point must lie on the λ_0 vs ω curve. If the λ_0 vs ω curve for the $t(x)$ distribution is similar to that of the reference panel, the critical value of λ_0 for the $t(x)$ must be greater than or equal to the numerical value λ_e chosen

$$\lambda_{cr} \geq \lambda_e$$

If this assumption is valid, a procedure can be formulated to find a panel which satisfies the optimality equations and also has a value of λ_{cr} identical to that of the reference panel. It is as follows:

a) Set λ_0 in the optimality equations equal to 343.2 and choose ω such that $\omega_1 < \omega < \omega_2$.

b) Solve the optimality equations. The resulting panel should have a value λ_{cr} which is greater than or equal to 343.2. Evaluate J.

c) Keep λ_0 set at 343.2. Change ω slightly from that chosen in part a. Repeat part b. Compare values of J. Change ω again in such a way as to decrease J.

d) Repeat parts a, b, and c to find J as a function of ω. The point where $dJ/d\omega = 0$ will be the least weight design and have a value of λ_{cr} exactly equal to 343.2. Intuitively, if the problem is properly posed, J must have at least one minimum value.

A Numerical Solution Scheme

To find the unknown function $t(x)$, one must solve a set of eight simultaneous, first-order, nonlinear differential equations. These equations have eight specified boundary conditions, divided equally between two boundary points, $x = 0$ and $x = 1$. Several algorithms exist for the numerical solution of such two-point boundary value problems. Among these methods are the so-called "shooting techniques." These techniques require the estimation of the unknown or unspecified boundary conditions at one boundary. These estimations are used, together with the specified conditions, to provide initial or starting values for numerical integration schemes. The equations are then numerically integrated to obtain values of the specified variables at the other boundary point. A comparison of the values obtained in this manner with those specified in the problem statement determines the accuracy of the estimate of the undetermined initial conditions. If the calculated final values differ greatly from those required, new estimates of the undetermined parameters are necessary.

The generation of better estimates of the undetermined boundary conditions has been the subject of many papers in the optimal control area. One such algorithm developed is the "transition matrix" method.[4] Application of this method to other aeroelastic optimization problems may be found in Refs. 3 and 6–8. The method itself will be described briefly here.

Let us be given a set of $2N$ simultaneous first-order differential equations of the form

$$dy_i/dx = f_i(y_j,t) \qquad i = 1,2, \ldots 2N \qquad (20a)$$
$$x_0 \leq x \leq x_f$$

and the control equation

$$t(x) = g(y_i) \qquad (20b)$$

together with N specified boundary conditions at x_0 of the form

$$y_j(x_0) = C_j \qquad (20c)$$

and N boundary conditions at x_f

$$y_i(x_f) = C_i \tag{20d}$$

C_i and C_j refer to constant values prescribed by the problem.

To start the numerical integration scheme, N undetermined values of the functions $y_j(x)$ are estimated at x_0. The algorithm logic is as follows:

a) Integrate the $2N$ equations numerically from x_0 to x_f. The control equation (20b) is used to calculate $t(x)$ as the integration progresses.

b) The values of $y_i(x_f)$ are recorded. These recorded values of $y_i(x_f)$ are compared to those values specified in Eq. (20d). In general, these values will not correspond. The difference between the numerical values and the required values is denoted as

$$\{Q_i\} = \{y_i(x_f) - C_i\} \tag{21}$$

c) A change in an estimated value of $y_j(x_0)$ will cause a change in $\{Q_i\}$. The partial derivatives $\partial Q_i / \partial y_j(x_0)$ may be calculated either numerically or using unit perturbation solutions.[4] This computation for structural problems is discussed in Refs. 3, 7, and 8. An array or matrix of values of these partial derivatives is then formed. This $N \times N$ square matrix is called the transition matrix, TR.

d) The transition matrix is the first-order approximation between *changes* in estimated boundary conditions at x_0 and *changes* in the differences between calculated values and required values at x_f, denoted as $\{\Delta Q_i\}$

$$\{\Delta Q_i\} = [TR]\{\Delta y_j(x_0)\} \tag{22}$$

If the transition matrix can be inverted, then the changes $\{\Delta y_j(x_0)\}$ can be found as a function of $\{\Delta Q_i\}$.

e) Let the change, $\{\Delta Q_i\}$, be some negative fraction of the vector $\{Q_i\}$

$$\{\Delta Q_i\} = -\varepsilon\{Q_i\} \qquad 0 < \varepsilon \leq 1$$

Then

$$\{\Delta y_j(x_0)\} = -\varepsilon[TR]^{-1}\{Q_i\} \tag{23}$$

The constant ε is a measure of the linearity of the relation between $\{\Delta y_j(x_0)\}$ and $\{\Delta Q_i\}$. If the unknown initial conditions at x_0 are changed by an amount given in Eq. (23) and the optimality equations are integrated as in a, the result should be to drive the numerical values of $y_i(x_f)$ closer to C_i and to drive $\{Q_i\}$ to zero.

An iterative scheme can be set up using the steps outlined above. The iteration is terminated when the calculated values are near enough to those required in the problem statement. A tolerance must be provided to determine when to terminate iteration.

Optimal Thickness Solution

As the problem is presently formulated, one must solve simultaneously for both the state variables and the multipliers. The first attempts at solving the problem in this manner set λ_0 equal to 343.2 and set ω equal to the flutter frequency of the reference panel. The transition matrix algorithm was tried, but failed to work. To use the transition matrix algorithm, the equations must be integrated from $x = 0$ to $x = 1$. A wide range of combinations of values of estimated values of the undetermined boundary conditions $p(0)$, $r(0)$, $\lambda_w(0)$, and $\lambda_q(0)$ were tried. In no case did the integration procedure proceed beyond $x = 0.45$. In a typical trial solution, the value of $t(x)$ first increased monotonically from $x = 0$ to near $x = 0.35$. It then declined rapidly until it became negative near $x = 0.45$. These initial results were puzzling since Turner's finite element model[2] indicated that $t(x)$ should have a maximum near $x = 0.50$. The transition matrix method could not be made to work properly and was temporarily abandoned.

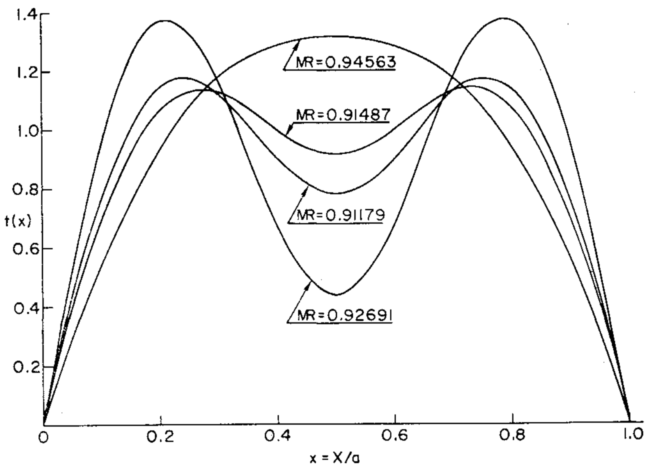

Fig. 3 Nondimensional panel thickness distributions with $t(x) = a \sin \pi x + b \sin 3\pi x$ which satisfies the constraint equations for $\delta_1 = 0.7$, $z_0 = 1.82$, $\lambda_0 = 343.2$.

An approximate method, where the function $t(x)$ is modeled as a truncated sine series, is developed in Ref. 8. This work found only symmetric $t(x)$ functions. A typical set of distributions found from that study is shown in Fig. 3. Because of the initial experience with the transition matrix method and with the sine series approximation, it was postulated that the optimal thickness distribution is, in fact, symmetric about $x = 0.5$. Although no rigorous proof can be given, it seems extremely unlikely, on the basis of numerical evidence, that a nonsymmetric optimal distribution exists. Therefore, for the remainder of this paper, symmetry of $t(x)$ will be assumed. This means that

$$t(x) = t(1 - x) \tag{24}$$

Reference 8 shows that, for a fixed frequency of free vibration of a conservative system, certain multiplier functions and certain state variable equations satisfy the same differential equations to within a multiplicative constant. For these problems

$$\{g_i\} = A\{\lambda_i\} \tag{25}$$

where g_i is the ith state variable and A is a real matrix.

The panel equations describe a nonconservative system. Nevertheless, a relation similar to Eq. (25) exists if Eq. (24) is true. This relation is, in matrix form,

$$\begin{Bmatrix} \lambda_w(x) \\ \lambda_p(x) \\ \lambda_q(x) \\ \lambda_r(x) \end{Bmatrix} = \frac{1}{B} \begin{bmatrix} \lambda_0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} w(\xi) \\ p(\xi) \\ q(\xi) \\ r(\xi) \end{Bmatrix} \tag{26}$$

where $\xi = 1 - x$.

If the relations in Eq. (26) are substituted into the Lagrange multiplier equations, the constraint equations as a function of the argument ξ result.

The constant B is a real number. This constant occurs because the problem has an eigenvalue constraint and therefore involves mode shapes which are undetermined to a constant factor. New problem variables may be defined which permit a sizable reduction in the numerical solution of the problem. These new variables are functions of the argument ξ

$$\begin{aligned} W(\xi) &= w(1 - x) \\ P(\xi) &= p(1 - x) \qquad 0 \leq \xi \leq 1 \\ Q(\xi) &= q(1 - x) \qquad 0 \leq x \leq 1 \\ R(\xi) &= r(1 - x) \end{aligned}$$

These new functions of ξ may be substituted into the state variable equations. The result is a set of differential equations in W, P, Q, and R

$$\dot{W}(\xi) = -P(\xi) \quad (27a)$$
$$\dot{P}(\xi) = -Q(\xi)/t(\xi) \qquad 0 \leq \xi \leq 1 \quad (27b)$$
$$\dot{Q}(\xi) = -R(\xi) \quad (27c)$$
$$\dot{R}(\xi) = -[\alpha t(\xi) + \beta]W(\xi) + \lambda_0 P(\xi) \quad (27d)$$

The notation $(\dot{\cdot})$ refers to differentiation with respect to ξ. The boundary conditions are

$$W(0) = W(1) = Q(0) = Q(1) = 0$$

With these new variables, and the relations in Eq. (26), the expression for the optimal thickness distribution becomes

$$t(x) = t(\xi) = \begin{cases} t_1(x); \text{ if } t_1 > t_{\min} \\ t_{\min}; \text{ if } t_1 \leq t_{\min} \end{cases}$$

where $t_1(x)$ is given by

$$t_1^2(x) = t_1^2(\xi) = Bq(x)Q(\xi)/[1 + \alpha Bw(x)W(\xi)] \quad (28)$$

An examination of Eq. (28) reveals that the constant B depends on the behavior of the state variable function $q(x)$. Because of the boundary conditions, $t(0)$ and $t(1)$ are zero if no minimum thickness constraint is enforced. If $q(0^+)$ and $Q(0^+)$ are of opposite sign, then B must be a negative real number if $t^2(0^+)$ is to be non-negative. The flutter modal bending moment for the reference panel does in fact have this behavior. Similarly, if B is to be a positive constant, $q(0^+)$ and $Q(0^+)$ must have the same signs.

The assumption of panel symmetry does not really reduce the number of variables in the problem, because new functions W, P, Q, and R have been introduced. Panel symmetry does, however, reduce the amount of numerical computation by one-half. If $w(x)$, $p(x)$, $q(x)$, $r(x)$ and $W(\xi)$, $P(\xi)$, $Q(\xi)$, $R(\xi)$ are taken as the problem unknowns, coupled together by Eq. (28), then, at $x = \xi = \frac{1}{2}$, continuity must be enforced. That is

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} w(1/2) \\ p(1/2) \\ q(1/2) \\ r(1/2) \end{Bmatrix} - \begin{Bmatrix} W(1/2) \\ P(1/2) \\ Q(1/2) \\ R(1/2) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (29)$$

The transition matrix procedure can be employed to enforce this continuity. The values of $p(0)$, $r(0)$, $P(0)$, $R(0)$ must be estimated. Equations (11a–d) together with Eqs. (27a–d and 28) can be solved numerically to find $\{Q_i\}$ in Eq. (29). If the resulting column matrix $\{Q_i\}$ is not sufficiently close to zero, then a transition matrix can be calculated to find new estimates of $p(0)$, $r(0)$, $P(0)$, $R(0)$ to force $\{Q_i\}$ closer to zero. This means that eight nonlinear equations are integrated from zero to one-half instead of zero to unity. The saving in computer time is substantial.

It is still difficult to obtain starting values of $p(0)$, $r(0)$, $P(0)$, $R(0)$ which permit the numerical solution to progress from zero to one-half. This problem can be remedied by calculating "intermediate solutions." The λ_0 vs ω (or z_0) curve is composed of two "branches." One branch defines the behavior of the fundamental frequency of the reference panel as a function of λ_0, while the second branch gives the behavior of the second natural frequency. Both frequency branches merge at λ_{cr}.

Theoretically, for any choice of λ_0 and ω (or z_0), the optimality equations can be solved for a thickness distribution and the undetermined parameters $p(0)$, $r(0)$, $P(0)$, and $R(0)$. Thus, these parameters are functions of λ_0 and ω. If λ_0 is zero, then the optimality equations are those for free vibration in a specified mode. From Eq. (28), if the vibration mode is symmetric then B is positive, while B must be negative for a nonsymmetric mode of vibration.

Reference 8 discusses the free vibration beam optimization problem for both fixed first and second frequencies. The values of the undetermined parameters are tabulated for various values of δ_1. These parameters may be used as estimates for a problem for which λ_0 is slightly greater than zero. The strategy is to move up a branch of the reference panel λ_0 curve by taking advantage of the knowledge of the behavior of the undetermined parameters as functions of λ_0 and ω. The solutions for which λ_0 is greater than zero, but less than 343.2, are called intermediate solutions. It is found that the convergence of the transition matrix method is rapid since interpolation from previous solutions provides close estimates to succeeding problems.

It is interesting to note that this strategy does not work on the first mode branch but does work on the second mode branch. In the vicinity of λ_0 equal to 50 on the first branch, the solution convergence is poor. At this point, the optimal $t(x)$ distribution has a maximum value at $x = 0.5$. Evidently the solution tries to change its behavior and the transition matrix method fails.

The move up the second branch is extremely easy. The values of the undetermined parameters are well-behaved functions of λ_0 and ω and are thus easy to plot and extrapolate. The only difficulty one encounters is the tendency of some state variables to become small in comparison to others. This leads to numerical errors in the integration scheme. This can be overcome either by scaling the variables or by resorting to simple solution schemes such as Euler's method. Since the solution interval length is short, a small step size can be used to ensure accuracy.

The result of the above strategy is to arrive at a solution to the optimality equations for which λ_0 is equal to 343.2. The second part of the solution method is then to change ω (or z_0) and determine changes in the objective function. Figure 4 shows the "mass ratios" of optimum panels which satisfy the constraint equations for the same value of λ_0 (343.2), but have differing values of the frequency parameter z_0. The mass ratio is the ratio between the total weight of the optimum panel to the total weight of the reference panel. In terms of the objective function, the mass ratio is given by

$$MR = \delta_1 J + (1 - \delta_1)$$

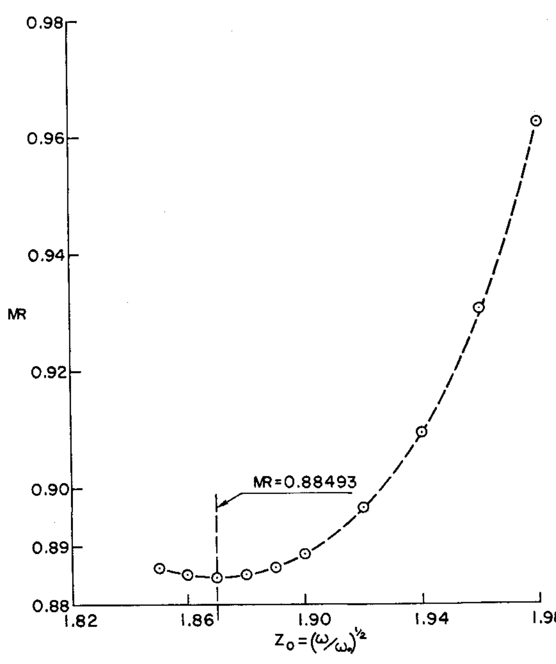

Fig. 4 Mass ratio vs z_0 for an optimum panel with $\lambda_0 = 343.2$, $\delta_1 = 0.7$, and $t_{\min} = 0.10$.

The mass ratio for a panel with δ_1 equal to 0.7 is found to be $MR = 0.885$. The flutter frequency parameter ratio z_0 is approximately 1.87, slightly higher than that of the reference panel. This panel is shown in Fig. 5. A minimum thickness constraint of 0.10 has been imposed. Figure 6 shows the mode shape and the modal bending moment associated with this panel if B is set equal to negative unity. Figure 7 shows several optimum panel thickness distributions with the same value of t_{min} but with different values of δ_1.

The parameter δ_1 was originally included to ensure that the problem was properly posed. If λ_0 is equal to zero, the optimality equations are such that a trivial or zero thickness solution can occur. The natural frequency in bending involves a ratio between the plate bending stiffness and the mass per unit area. Since both quantities are linear functions of the thickness for a sandwich panel, the dependence of frequency on thickness disappears if nonstructural mass is not included. Theoretically, this allows the panel to vanish. Since this is not physically reasonable, nonstructural mass is added to ensure the dependence of frequency on thickness and to keep the problem properly posed. If λ_0 is nonzero, no such trivial solution can occur because the panel must resist aerodynamic forces in addition to inertia loads. Figure 8 shows an optimal thickness distribution for a panel with δ_1 equal to unity and t_{min} equal to one-half. Note that the flutter frequency is close to that of the reference panel.

The optimal thickness distributions reflect the fact that the flutter mode has a substantial contribution from the second bending mode. The panel shown in Fig. 5 was analyzed for flutter by calculating a complete λ_0 curve and verifying that the first and second flexural frequencies did, in fact, merge at λ_{cr} equal to 343.2. Figure 2 shows these results compared to those for the reference panel. Subsequent finite element model studies in Ref. 8 verified the validity of these results.

It is interesting to compare the panel flutter optimization results to another similar optimization problem. If either the first or second natural frequencies of the plate-beam are held constant during weight optimization, then for $\delta_1 = 0.7$ and $t_{min} = 0.10$, the mass ratio is

$$MR = 0.9114 \text{ (1st or 2nd frequency fixed)}$$

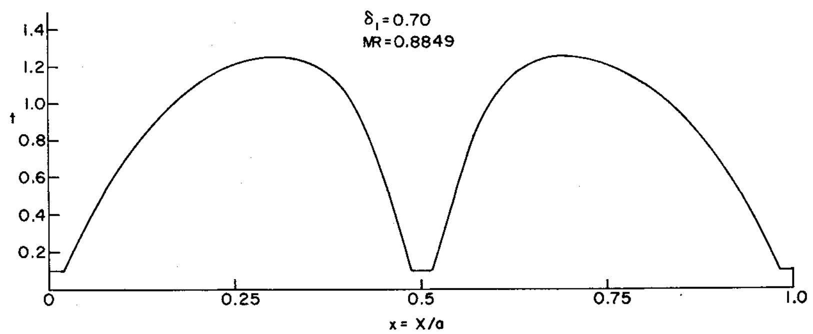

Fig. 5 Optimal thickness parameter distribution for a panel with $\lambda_{cr} = 343.20$, $\delta_1 = 0.70$, $t_{min} = 0.10$.

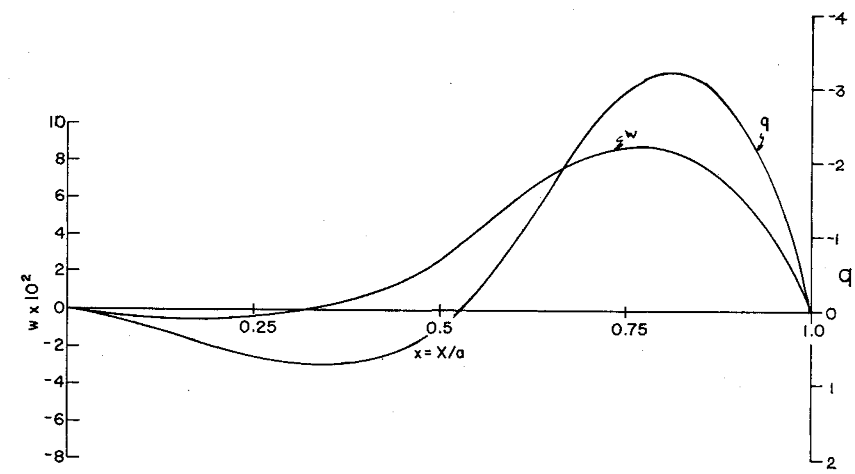

Fig. 6 Nondimensional deflection and bending moment for panel shown in Fig. 5; $\lambda_{cr} = 343.20$, $\delta_1 = 0.70$, $t_{min} = 0.10$.

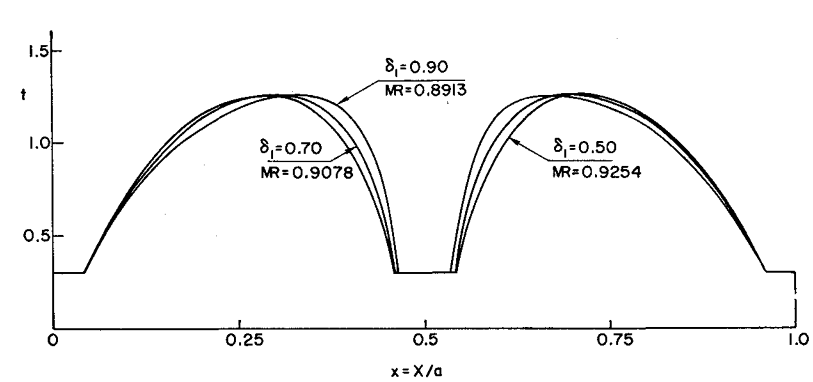

Fig. 7 Effect of varying δ_1; $\lambda_{cr} = 343.20$.

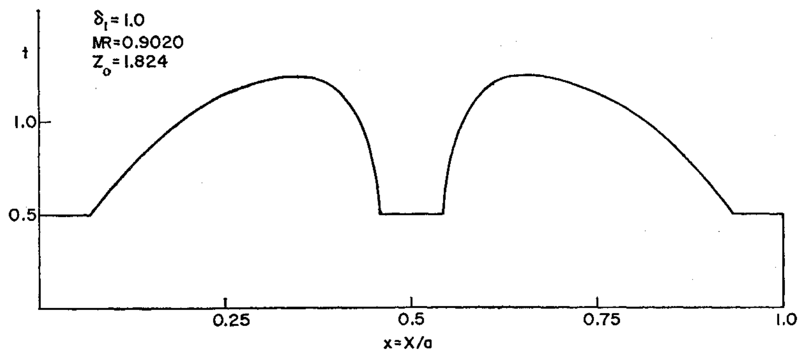

Fig. 8 Optimal thickness parameter distribution; $\lambda_{cr} = 343.20$, $\delta_1 = 1.0$, $t_{min} = 0.50$.

For the same structure, if only the flutter parameter is fixed, the mass ratio is smaller.

$$MR = 0.8849 \text{ (flutter parameter fixed)}$$

Conclusions

The results of this study show that control theory methods provide valuable tools for the solution to some aeroelastic optimization problems. The results presented here are, of course, the result of a highly idealized model. The panel model used is far too simple to be of practical use. However, the results do define a classical solution to an aeroelastic optimization problem and provide insight into the optimization process itself. They show, for instance, that the flutter mode shape of the optimum weight panel does not differ drastically from that of the reference panel. The numerical work also seems to preclude the possibility that a nonsymmetric optimal thickness solution exists. This would mean that the symmetric solution shown to exist by Turner is the only solution to the problem.

The weight reduction of nearly 12% shows that meaningful reductions in panel weight are feasible. The symmetrical optimal distribution of thickness is also useful. If one were to stiffen a panel against flutter, it seems that an addition of structural weight at the one-third chord position is as effective as an addition at the two-thirds chord position. The worst positions to add weight are at the panel edges and at the midchord position.

It would be interesting to investigate the effect of aerodynamic damping on the optimum panel weight and thickness distribution. However, this would double the number of necessary equations and thus make the solution difficult. Also, a two-dimensional panel model which has a finite span would be more realistic than that used in this study. The techniques necessary to do such two-dimensional plate, free-vibration, optimization problems have recently been studied by Armand.[9] These same techniques could prove valuable for the flutter problem.

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